Biarc Curve Fitting

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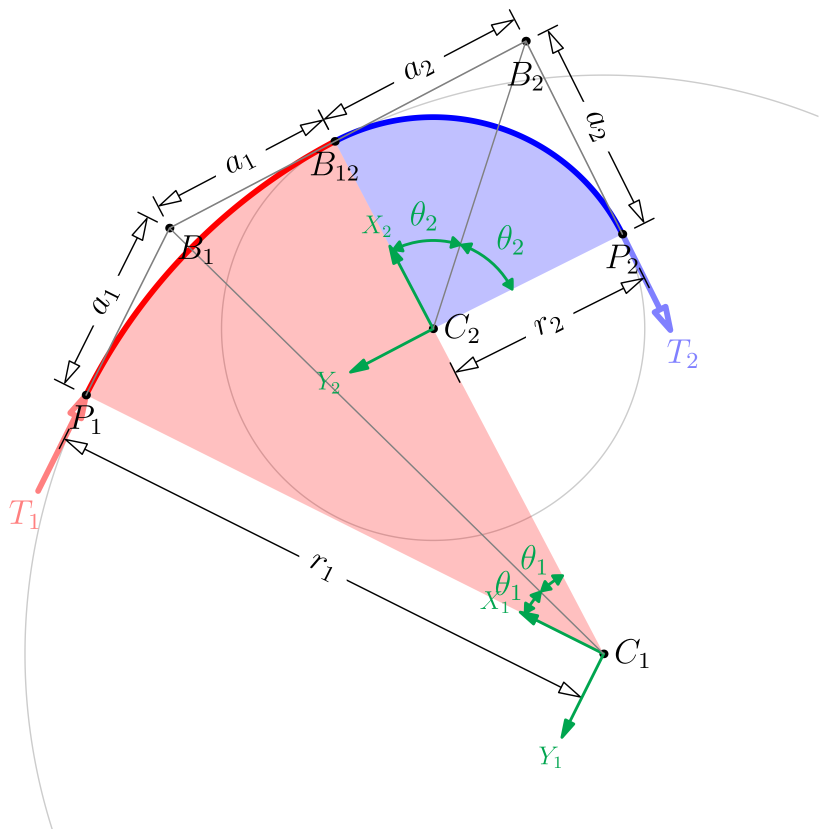
1 Introduction

Biarc curve fitting determines two circular arcs passing through two given points and tangents at those points. When applied to a series of points, it determines a piecewise circular arc interpolation of given points.

This document summarizes the work of [ROSSIGNAC,REQUICHA]

2 Formulation of Jaroslaw & Aristides

Following figure depicts a biarc curve fitting through points $\mathbf{P}_1, \mathbf{P}_2$ with given tangents $\mathbf{T}_1, \mathbf{T}_2$.



apex points of two segments are given by

$$\mathbf{B}_1 = \mathbf{P}_1 + a_1\mathbf{T}_1$$
$$\mathbf{B}_2 = \mathbf{P}_2 - a_2\mathbf{T}_2$$

relations that those points must satisfy are

$$|\mathbf{B}_1 - \mathbf{P}_1| = |\mathbf{B}_{12} - \mathbf{B}_1| = a_1 \quad (1)$$
$$|\mathbf{B}_2 - \mathbf{P}_2| = |\mathbf{B}_{12} - \mathbf{B}_2| = a_2 \quad (2)$$
$$|\mathbf{B}_2 - \mathbf{B}_1| = a_1 + a_2 \quad (3)$$

where junction points of two arcs is

$$\mathbf{B}_{12} = \mathbf{B}_1 + \frac{a_1}{a_1 + a_2}(\mathbf{B}_2 - \mathbf{B}_1) = \frac{a_2\mathbf{B}_1 + a_1\mathbf{B}_2}{a_1 + a_2}$$

Substituting definitions of $\mathbf{B}_1, \mathbf{B}_2$ in basic equation

$$\begin{aligned}\mathbf{B}_2 - \mathbf{B}_1 &= (\mathbf{P}_2 - a_2\mathbf{T}_2) - (\mathbf{P}_1 + a_1\mathbf{T}_1)\\ &= \mathbf{P}_2 - \mathbf{P}_1 - (a_1\mathbf{T}_1 + a_2\mathbf{T}_2)\end{aligned}$$

and defining

$$\mathbf{P}_2 - \mathbf{P}_1 \equiv \mathbf{S}$$

basic equation becomes

$$\begin{aligned}|\mathbf{B}_2 - \mathbf{B}_1|^2 &= (a_1 + a_2)^2\\ (\mathbf{P}_2 - \mathbf{P}_1 - (a_1\mathbf{T}_1 + a_2\mathbf{T}_2))^2 &= (a_1 + a_2)^2\\ (\mathbf{S} - (a_1\mathbf{T}_1 + a_2\mathbf{T}_2))^2 &= (a_1 + a_2)^2\\ \mathbf{S}^2 - 2\mathbf{S}\cdot(a_1\mathbf{T}_1 + a_2\mathbf{T}_2) + (a_1\mathbf{T}_1 + a_2\mathbf{T}_2)^2 &= (a_1 + a_2)^2\\ \mathbf{S}^2 - 2\mathbf{S}\cdot(a_1\mathbf{T}_1 + a_2\mathbf{T}_2) + a_1^2\mathbf{T}_1^2 + 2a_1a_2\mathbf{T}_1\cdot\mathbf{T}_2 + a_2^2\mathbf{T}_2^2 &= a_1^2 + 2a_1a_2 + a_2^2\\ \mathbf{S}^2 - 2S\cdot a_1\mathbf{T}_1 - 2\mathbf{S}\cdot a_2\mathbf{T}_2 + a_1^2 + 2a_1a_2\mathbf{T}_1\cdot\mathbf{T}_2 + a_2^2 &= a_1^2 + 2a_1a_2 + a_2^2\\ \mathbf{S}^2 - 2\mathbf{S}\cdot a_1\mathbf{T}_1 - 2\mathbf{S}\cdot a_2\mathbf{T}_2 + 2a_1a_2\mathbf{T}_1\cdot\mathbf{T}_2 &= 2a_1a_2\end{aligned}$$

$$a_1a_2(\mathbf{T}_1\cdot\mathbf{T}_2 - 1) + \frac{\mathbf{S}^2}{2} = a_1\mathbf{S}\cdot\mathbf{T}_1 + a_2\mathbf{S}\cdot\mathbf{T}_2$$

General formula is given (in [ROSSIGNAC,REQUICHA, p. 300])

$$a_2 = \frac{a_1(\mathbf{S}\cdot\mathbf{T}_1) - \frac{1}{2}\|\mathbf{S}\|^2}{a_1(\mathbf{T}_1\cdot\mathbf{T}_2 - 1) - \mathbf{S}\cdot\mathbf{T}_2}$$

2.0.1 Specified ratio of a_i

If the ratio of two side lengths is specified

$$\rho = \frac{a_2}{a_1}$$

then

$$a_1\rho a_1(\mathbf{T}_1\cdot\mathbf{T}_2 - 1) + \frac{\mathbf{S}^2}{2} = a_1\mathbf{S}\cdot\mathbf{T}_1 + \rho a_1\mathbf{S}\cdot\mathbf{T}_2$$

$$a_1^2[\rho(\mathbf{T}_1\cdot\mathbf{T}_2 - 1)] - a_1(\mathbf{S}\cdot\mathbf{T}_1 + \rho\mathbf{S}\cdot\mathbf{T}_2) + \frac{\mathbf{S}^2}{2} = 0$$

this general relation has some special cases that must be handled. These cases are described in [ROSSIGNAC,REQUICHA]

2.1 Other Relations Between Auxiliary Points

When junction point expressed in initial variables

$$\begin{aligned}\mathbf{B}_{12} &= \frac{a_2\mathbf{B}_1 + a_1\mathbf{B}_2}{a_1 + a_2} \\ &= \frac{a_2\left(\mathbf{P}_1 + a_1\mathbf{T}_1\right) + a_1\left(\mathbf{P}_2 - a_2\mathbf{T}_2\right)}{a_1 + a_2} \\ &= \frac{a_2\mathbf{P}_1 + a_2a_1\mathbf{T}_1 + a_1\mathbf{P}_2 - a_1a_2\mathbf{T}_2}{a_1 + a_2} \\ &= \frac{a_2\mathbf{P}_1 + a_1\mathbf{P}_2 + a_2a_1\left(\mathbf{T}_1 - \mathbf{T}_2\right)}{a_1 + a_2}\end{aligned}$$

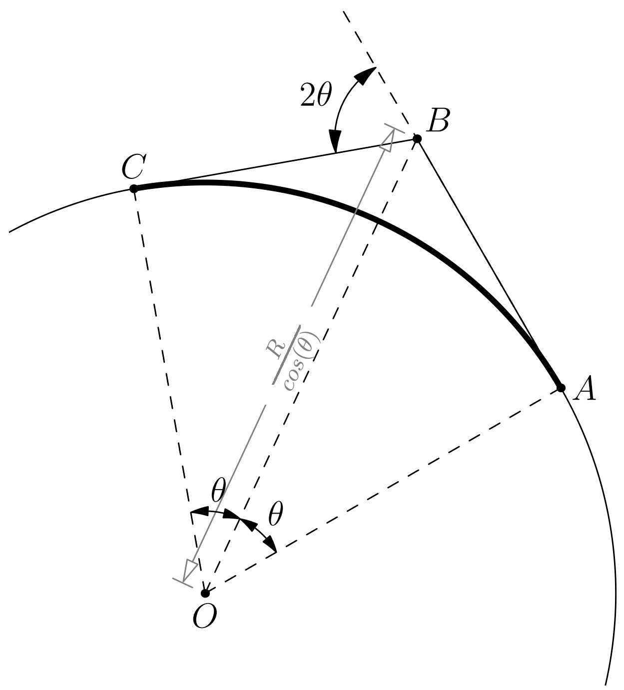
chords of arcs become

$$\begin{aligned}\mathbf{B}_{12} - \mathbf{P}_1 &= \frac{a_2\mathbf{P}_1 + a_1\mathbf{P}_2 + a_2a_1\left(\mathbf{T}_1 - \mathbf{T}_2\right)}{a_1 + a_2} - \mathbf{P}_1 \\ &= \frac{a_2\mathbf{P}_1 + a_1\mathbf{P}_2 + a_2a_1\left(\mathbf{T}_1 - \mathbf{T}_2\right) - \left(a_1 + a_2\right)\mathbf{P}_1}{a_1 + a_2} \\ &= \frac{a_1\left(\mathbf{P}_2 - \mathbf{P}_1\right) + a_2a_1\left(\mathbf{T}_1 - \mathbf{T}_2\right)}{a_1 + a_2}\end{aligned}$$

$$\begin{aligned}\mathbf{B}_{12} - \mathbf{P}_2 &= \frac{a_2\mathbf{P}_1 + a_1\mathbf{P}_2 + a_2a_1\left(\mathbf{T}_1 - \mathbf{T}_2\right)}{a_1 + a_2} - \mathbf{P}_2 \\ &= \frac{a_2\mathbf{P}_1 + a_1\mathbf{P}_2 + a_2a_1\left(\mathbf{T}_1 - \mathbf{T}_2\right) - \left(a_1 + a_2\right)\mathbf{P}_2}{a_1 + a_2} \\ &= \frac{a_2\left(\mathbf{P}_1 - \mathbf{P}_2\right) + a_2a_1\left(\mathbf{T}_1 - \mathbf{T}_2\right)}{a_1 + a_2}\end{aligned}$$

2.2 Computing arc center and angle

This section describes the computation of center and arc-angle of two arcs from given two points on circle $(\mathbf{A}, \mathbf{C})$ and apex point $\mathbf{B}$. Consider following figure



If points $\mathbf{A}, \mathbf{C}$ are on circle and tangents to circle at $\mathbf{A}, \mathbf{C}$ intersect at apex $\mathbf{B}$, then

$$\overrightarrow{AB} \cdot \overrightarrow{BC} = \cos(2\theta) = D$$
$$\overrightarrow{AB} \times \overrightarrow{BC} = \sin(2\theta)\,\mathbf{a}$$

(where $\mathbf{a}$ is arc-plane normal).

From

$$\cos(2\theta) = 2\cos^2\theta - 1$$
$$= 1 - 2\sin^2\theta$$

$$\cos\theta = \sqrt{\frac{1+D}{2}}$$
$$\sin\theta = \sqrt{\frac{1-D}{2}}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \sqrt{\frac{1-D}{1+D}}$$

from

$$\tan\theta = \frac{\overline{AB}}{r}$$

radius is determined as

$$r = \frac{\overline{AB}}{\tan\theta} = \sqrt{\frac{1+D}{1-D}}\overline{AB}$$

from

$$\cos\theta = \frac{r}{\overline{BO}}$$

circle center-to-apex distance BO is computed as

$$\overline{BO} = \frac{r}{\cos\theta}$$

then center of arc is

$$\mathbf{O} = \mathbf{B} + \frac{r}{\cos\theta}\frac{\overrightarrow{BA} + \overrightarrow{BC}}{\left|\overrightarrow{BA} + \overrightarrow{BC}\right|}$$

2.2.1 Using complex numbers

Center can be found from the intersection of line segments through point a and c with directions $i(b-a), i(c-b)$ that is

$$a + \lambda i(b-a) = c + \mu i(c-b)$$

Placing P_1, P_2 on real axis with midpoint $\frac{P_1+P_2}{2}$ in origin, start and points become

$$p_1 = -\frac{s}{2} + 0i$$
$$p_2 = +\frac{s}{2} + 0i$$

junction points is

$$b_{12} = \frac{a_2\frac{-s}{2} + a_1\frac{s}{2} + a_2a_1\left(e^{i\theta_1} - e^{i\theta_2}\right)}{a_1 + a_2}$$

and apex points are

$$
\begin{aligned}
b_1 &= -\frac{s}{2} + a_1 e^{i\theta_1} \\
b_2 &= +\frac{s}{2} - a_2 e^{i\theta_2}
\end{aligned}
$$

For the first control triangle $\left(-\frac{s}{2}, -\frac{s}{2} + a_1 e^{i\theta_1}, \frac{a_2 \frac{-s}{2} + a_1 \frac{s}{2} + a_2 a_1 \left(e^{i\theta_1} - e^{i\theta_2}\right)}{a_1 + a_2}\right)$ intersection is described by

$$
\begin{aligned}
-\frac{s}{2} + \lambda i e^{i\theta_1} &= +\frac{s}{2} + \mu i e^{i\theta_2} \\
\lambda i e^{i\theta_1} - \mu i e^{i\theta_2} &= s \\
\lambda e^{i\theta_1} - \mu e^{i\theta_2} &= -is
\end{aligned}
$$

$$
\begin{aligned}
\lambda \cos\theta_1 - \mu \cos\theta_2 &= 0 \\
\lambda \sin\theta_1 - \mu \sin\theta_2 &= -s
\end{aligned}
$$

$$
\begin{aligned}
\lambda \cos\theta_1 \sin\theta_1 - \mu \cos\theta_2 \sin\theta_1 &= 0 \\
\lambda \sin\theta_1 \cos\theta_1 - \mu \sin\theta_2 \cos\theta_1 &= -s \cos\theta_1
\end{aligned}
$$

$$
-\mu \left(\cos\theta_2 \sin\theta_1 - \sin\theta_2 \cos\theta_1\right) = s \cos\theta_1
$$

$$
\begin{aligned}
\mu &= \frac{s \cos\theta_1}{\cos\theta_2 \sin\theta_1 - \sin\theta_2 \cos\theta_1} \\
&= \frac{\cos\theta_1}{\cos\left(\theta_1 + \theta_2\right)} s
\end{aligned}
$$

References

[ROSSIGNAC,REQUICHA] "*Piecewise circular curves for geometric modeling*", Jaroslaw R. Rossignac, Aristides A.G. Requicha, IBM J. RES. DEVELOP. VOL.3, NO.3 May 1987